<u>"Exploration of digit patterns in outcomes of selected calculations - for example *80 ÷ 81*"</u>

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<u>Trigger</u>

In times past the presenter spotted that the computation ***80 ÷ 81*** on a commonly-available calculator display yielded the output

80 ÷ 81 *= 0.987654321* ..... see later for detail.

The pattern of digits in this output was considered to be "intriguing" and, maybe, worthy of exploration.

<u>Approach</u>

We will be looking at repeated patterns of digits and of blocks of digits recurring within decimal representations of selected real numbers. Aiming to have little in the way of advanced mathematical content to distract us, our intent is to focus attention explicitly on particular skills associated with "exploration" which can often prove to be of "added value" within more formal mathematical settings.

<u>General Overview</u>

Think first in terms of an investigation of a case in a fictional "who-dunnit" story featuring an amateur super-sleuth – for example a Sherlock Holmes, a Jane Marple, or an Hercule Poirot. Our super-sleuth is the first to "spot" a potential key clue, very likely one possibly overlooked by others. As the story unfolds, the super-sleuth goes on to "follow-through / explore" the clue, to select an appropriate "interpretation", and finally to proceed speedily to a triumphant "conclusion" of the case.

<u>Aim</u>

The aim of this presentation is to plant in your minds the seeds of the usefulness of the sleuth's skill of independently, almost incidentally, "spotting" potential key clues and "exploring" these in a mathematical context. Depending on the context, the outcomes can include (i) the illumination and validation of your current understanding of an established solution, through (ii) the discovery of a more elegant or time-saving approach to an established solution, or (iii) one of no gain, but in such circumstances you might convince yourself that you have learned something from the experience !

<u>Context</u>

Here we are to look at examples of "spotting" key patterns in arrays of digits in calculator outputs for selected computations, speculating as to possible relationships, and then exploring them further.

More than likely, but not always, our exploration may prove to be inconclusive in-so-far-as our speculated relationships might all disintegrate or otherwise turn out to be invalid.

Unlike many other Maths Club sessions, this presentation will leave several loose-ends and unresolved issues. Regard these as possible candidates for future independent explorations of your own.

Preliminary **Examples - inverses of selected single-digit and two-digit integer values**

Repeating digits provide one type of spottable patterns of digits in decimal representations of inverses of integer values.

Calculation	Result	Terminating	Repeating Digits(s)
1 ÷ 1	*1.0*	✔	
1 ÷ 2	*0.5*	✔	
1 ÷ 3	*0.3333333333'*		*"3"*
1 ÷ 4	*0.25*	✔	
1 ÷ 5	*0.2*	✔	
1 ÷ 6	*0.1666666666'*		*"6"*
1 ÷ 7	*0.142857142857'*		*"142857"*
1 ÷ 8	*0.125*	✔	
1 ÷ 9	*0.111111111'*		*"1"*
1 ÷ 10	*0.1*	✔	
1 ÷ 11	*0.89090909090'*		*"90"*
1 ÷ 12	*0.8333333333'*		"3"
1 ÷ 13	*0.076923076923'*		*"076923"*

Notation: Digits underlined and dashed denote repeating digits or blocks of digits:
(i) *1'* denotes single digit *1* repeating (ii) *142857'* denotes block *142857* repeating.

Scope

Our session starts with a preliminary check on your familiarity with the number-storage facility within your calculators.

Thereafter the effects of simple operations on selected patterns of arrays of digits are noted and speculative relationships between the operations and the changes in patterns arising from the operations are explored.

"Rounding errors" feature here only in respect of the role they play in the presentation of data. The onward consideration of the accrual of rounding and/or relative errors in the outcomes of computations lies outside the scope of the session – maybe to be considered possibly in a future session.

Adopted Font Convention for distinguishing input and output displays

In these notes, input displays are shown as ***bold italic***, output displays as *non-bold italic.*

Our calculators' screen-modes are assumed to be set for "line inputs/outputs" which yield results such as:

2 ÷ 3 *= 0.6666666667*

Here is another example, already mentioned, yielding "intriguing" output pattern.

80 ÷ 81 *= 0.987654321* ..... more detail later.

Exploring a calculator

Of interest are the role and content of a typical calculator memory store

Participants' activity: in the table below, input in turn the seven calculations/familiar symbols shown and in each case we subtract an exact copy of the displayed result to reveal the hidden remainder

(*Note for readers post-presentation: At the time the participants derived the Results as shown below and these were then subsequently discussed*)

For each input, we note from the previously hidden, but now "Revealed", Remainder terms whether or not the extended tail digits for the Displayed Result *appear* to indicate an ongoing "Repeating" or "Non-repeating" pattern of displayed digits.

Calculation	Displayed Result	Revealed Remainder	Rep /Non-Repeating
2 ÷ 3	*0.6666666667*	-3.3334×10^{-11}	*"6" repeating*
1 ÷ 3	*0.3333333333*	3.3333×10^{-11}	*"3" repeating*
e	*2.718281828*	4.5904×10^{-10}	*non repeating*
π	*3.141592654*	-4.102×10^{-10}	*non repeating*
1 ÷ 9	*0.1111111111*	1.1111×10^{-11}	*"1" repeating*
1 ÷ 27	*0.03703703704*	-2.963×10^{-12}	*"037"repeating*
80 ÷ 81	*0.987654321*	-1.2346×10^{-11}	*??? see later*

In respect of the forward-projection of identifiable repeating digits or blocks of digits the presence of a rounded-up digit disrupts the ongoing pattern. From this point on, following their recognition either by "prior experience" or by a corresponding negative "revealed remainder", all rounded-up final digits in "displayed results" are to be to be recognised as such and then "rounded back down".

Digression – Adoption of Temporary Notation

Starting with the set of all integers, selected sub-sets can be defined as "multiples of 2", "multiples of 3", etc. For brevity the notations "mo2", "mo3", etc are adopted here as indicating that a number is a "multiple of 2", a "multiple of 3", etc. It is self-evident that "even" is the term in common usage corresponding to "mo2". At the time of presentation your reporter did not recall any term in common usage corresponding to "mo3".

Exploring Sudoku-type nonuples

A "nonuple" is an array of nine characters – as is a "triple" an array of three.

"Sudoku-type" nonuples here are arrays of the nine distinct digits, 1, 2, ... 9, in any order.

Participants' activity:
For the moment, represent the value of the nine-digit number 123456789 by 'A'.

Multiply 'A' by the factor 2 and note whether or not the resulting product is a Sudoku-type nonuple. Repeat for multiples of 'A' by factors 3, ... 9.

Multiplier	Product
1	*123456789*
2	*246913578*
3	*370370370*
4	*493827156*
5	*617283945*
6	*740740734*
7	*864197523*
8	*987654312*
9	*1111111101*

Note: (i) The multipliers which result in Sudoku-type nonuple products, namely, 1, 2, 4, 5, 7, 8, are all non "mo3"s, whereas (ii) those which do not result in Sudoku-type nonuple products are all "mo3"s .

Here is a Take-away Challenge": At the time of preparation of this material, the presenter has no idea as to why the pattern of the classification turns out as it does. Have you?

ce that those multipliers which result in Sudoku-type nonuple products are 1, 2, 4, 5, 7, 8, which are all non "mo3"s. Those which do not are all "mo3"s .

Here is a Take-away Challenge": At
the time of preparation of this material, the presenter has no idea as to why
the pattern of the classification turns out as it does. Have you?

<u>Exploration – some standard series theory</u>

Reference is to be made to some standard results relating to selected power-series.
The derivations of these results are provided on an accompanying summary sheet.

<u>Exploration – some exercises and related discussion points</u>

1. List, to 12 decimal places, the values of the six quotients:
   1 ÷ 7, 2 ÷ 7, ..., 6 ÷ 7.
   What do you spot about any missing digits?
   What do you spot about the orders of the arrays of the digits present?

2. Likewise, list to 12 decimal places, the values of the twelve quotients:
   1 ÷ 13, 2 ÷ 13, ..., 12 ÷ 13.
   What do you spot about the combinations and the orders of the arrays present?

3. Write: ***1 ÷ 9*** = $\frac{1}{9} = \frac{0.1}{0.9} = \frac{0.1}{1-0.1} = 0.1\left(\frac{1}{1-0.1}\right)$

We recognise this as generating a Geometric Series

$$0.1\ x\left(\frac{1}{1-0.1}\right) = 0.1\ x\ (\ 1 + (0.1) + \ (0.1)^2\ +\ (0.1)^3 +\ \ldots)$$

Deduce that ***1 ÷ 9 = 0.111111......***

4. From first table, we presume: ***1 ÷ 7 = 0.142857142857......***

Conjecture ***1 ÷ 7 = N*** *x* ***(1 + 10*** $^{-6}$ ***+ 10*** $^{-12}$ ***+ ......)*** where ***N*** is a six digit array.

Thus we require ***1 ÷ 7 = N*** *x* $\frac{1}{1-10^{-6}}$

Solving for ***N*** yields ***N = 0.143857*** and thus our presumption is confirmed.

5. Finally consider ***80*** ÷ ***81***. There are, at least, three ways forward.

(i) Write $\frac{80}{81} = \frac{0.80}{0.81} = \frac{0.80}{(0.9)^2} = \frac{0.80}{(1-0.1)^2}$

We recognise this as generating a Negative Binomial Series.

$\frac{0.80}{(1-0.1)^2}$ *=0.80 x (1 +* 2 x *0.1 + 3 x* 0.1^2 *+......) = 0.987654320......*

(ii) From our very first calculation, we presume that ***80 ÷ 81 = 0.987654321***

Conjecture ***80 ÷ 81 = N*** *x* ***(1 + 10*** $^{-9}$ ***+ 10*** $^{-18}$ ***+ ......)*** where ***N*** is a nine digit array.

Solving for ***N*** yields ***N = 0.987654320.*** This value corresponds to that in (i) and not to that of our presumption, which we conclude was incorrect.

(iii) Keep it simple. Resort to (very) long division. Note that the digit-by-digit operations to calculate ***N*** repeat after the nine (decimal fraction) digits of ***N = 0.987654320***

In Conclusion

Our initial calculator output was displayed as

80 ÷ 81 = 0.987654321

Whilst it was intriguing, because it is rounded, the nine (decimal fraction) digit array as shown is not repeating. What we now know, three ways, is that what repeats is

80 ÷ 81 = 0.987654320

What we have learned is that the final digit obtained from a calculator or software display not always be taken as exact. Of greater depth, we have had some practice in working with power series

Endpiece

Although from a different context, the following quote from T.S. Eliot's "Four Quartets" might be considered very appropriate to the above material:

"We shall not cease from exploration
And the end of all our exploring
Will be to arrive where it started
And know the place for the first time".

MATHS CLUB – " EXPLORATION .... – SOME POWER SERIES"

Binomial series : for $(1 + x)^n$ where n is an integer $\geq$ 0

n=1 $(1 + x)^1 = 1 + x$

n=2 multiply above by $(1 + x)$

$(1 + x)^2 = (1 + x)(1 + x)$

$= 1 + x + x + x^2 = 1 + 2x + x^2$

n=3 multiply above by $(1 + x)$

$(1 + x)^3 = (1 + x)(1 + 2x + x^2)$

$= 1 + 2x + x^2 + x + 2x^2 + x^3 = 1 + 3x + 3x^2 + x^3$

etc ……

Geometric series : $(1 – x)^{-1}$, where $–1 < x < 1$

Initially for any x, set

$S_k = 1 + x + x^2 + x^3 + …… + x^k$ **where k is an integer $\geq$ 0. Refer to this equation as (1).**

Multiply above by x

$xS_k = \quad x + x^2 + x^3 + …… \quad x^k + x^{k+1}$. **Refer to this equation as (2).**

Subtract (2) from (1), giving

$S_k – xS_k = 1 – x^{k+1}$,

ie $(1 – x)S_k = 1 – x^{k+1}$

Hence $S_k = \frac{(1 – x^{k+1})}{(1 – x)}$.

For $–1 < x < 1$, let k tend to infinity and S_k tend to S.

Thus $S = \frac{1}{(1 – x)} = (1 – x)^{-1}$.

So, when $–1 < x < 1$, we have $(1 – x)^{-1} = 1 + x + x^2 + x^3 + ……$

Negative binomial series : for $(1 – x)^m$ where $–1 < x < 1$ and m is an integer < 0

m= -1 $(1 – x)^{-1} = 1 + x + x^2 + x^3 + ……$ **where $–1 < x < 1$ per geometric series**

multiply above by $(1 – x)^{-1} = 1 + x + x^2 + x^3 + ……$

m= -2 $(1 – x)^{-1}(1 – x)^{-1} = (1 + x + x^2 + x^3 + ……)(1 + x + x^2 + x^3 + ……)$

$= 1 + x + x^2 + x^3 + ……$

$x + x^2 + x^3 + ……$

$x^2 + x^3 + ……$

$x^3 + …$

$+ ……$

So $(1 – x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ……$ **where $–1 < x < 1$**

By multiplying above $(1 – x)^{-1} = 1 + x + x^2 + x^3 + ……$ **we can similarly derive**

m=-3 $(1 – x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + ……$ **where $–1 < x < 1$**

Pascal's Triangle for coefficients in binomial series

With appropriate interpretation the coefficients of powers of x in the binomial expansions for $(1 + x)^n$ where n is an integer $\geq$ 0 can be read horizontally and those for $(1 – x)^m$ where m is an integer < 0 and $–1 < x < 1$ can be read downhill diagonally.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Entries in further rows can be computed as the sum of the two adjacent entries immediately above.